AN INVERSION FORMULA FOR SOME CONVOLUTION TRANSFORMS

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ABSTRACT
Inversion and convergence results are achieved for some convolution transforms whose kernel's bilateral Laplace transform is $[\prod_{k=1}^{\infty}(1-(s^2/a_k^2))]^{-1}$ where a_k are complex.

1. The convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt \tag{1.1}$$

where

$$\int_{-\infty}^{\infty} e^{-st}G(t)dt = \left[\prod_{k=1}^{\infty}(1-s^2/a_k^2)\right]^{-1} \equiv [F(s)]^{-1} \tag{1.2}$$

and $\Sigma|a_k|^{-2} < \infty$ was treated by J. Dauns and D. V. Widder [1] and the author [2] when $|\arg a_k| \leqq \psi < \pi/2$. It was remarked that for some special sequences $\{a_k\}$ for which $\pi/4 < |\arg a_k| < \pi/2$ the transforms satisfying (1.1) and (1.2) exist but the hardships in finding an inversion formula of Hirschman-Widder type are obvious since

$$\lim_{m\to\infty}\int_{-\infty}^{\infty} |P_{2m}(D)G(t)|\,dt = \lim_{m\to\infty}\int_{-\infty}^{\infty}|G_{2m}(t)|\,dt = \infty$$

where $P_{2m}(D) = \prod_{k=1}^{m}(1-a_k^{-2}D^2)$ and $D = \frac{d}{dx}$ (see remarks by J. Dauns and D. V. Widder [1, p. 442]).

We shall treat a rather large class in this paper that will contain those mentioned in [1, Section 5]; we shall use a modification of an inversion formula by C. Standish, that was proved only for a subclass of the variation diminishing transforms, instead of that used by [1].

As is usually done, we define the class of convolution transforms by restrictions on the sequences $\{a_k\}$.

Class $C(\rho)$. A transform belongs to class $C(\rho)$ if:

Received January 23, 1968.

(1) For some real λ and ρ, $\lambda > 0$, $0 < \rho < 1$ the relation $N(x) \sim \lambda x^\rho x \to \infty$ holds, where $N(x)$ is the number of a_k satisfying $|a_k|^2 \leqq x$;

(2) $|\arg a_k| \leqq \psi$ where $\psi < \pi/2$ for $0 < \rho \leqq \frac{1}{2}$ and $\psi < \pi/4\rho$ for $\frac{1}{2} < \rho < 1$. The inversion formula achieved is

$$(1.3) \quad \lim_{t \to 0+} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) d\xi \int_{-8}^{\infty} F(iu) \exp[-tu^2 + iu(x-\xi)] du = \phi(x) \text{ a.e.}$$

For convenience we shall assume $0 < |a_k| \leqq |a_{k+1}|$.

2. **Preliminary estimates of** $F(s)$ **and** $G(t)$. To find a lower estimate for $F(s)$ we shall prove first the following lemma.

LEMMA 2.1. *Let* $|\arg a| \leqq \psi < \pi/2$, $0 < A < 1$, *then*

$$(2.1) \qquad |1 - z^2/a^2|^2 \geqq 1 - 2\frac{|z|^2}{|a|^2} \cos(1-A) 2\left(\frac{\pi}{2} - \psi\right) + \frac{|z|^4}{|a|^4}$$

uniformly for z satisfying $\min_{n=1,3} \left|\arg z - \frac{n\pi}{2}\right| \leqq A\left(\frac{\pi}{2} - \psi\right)$.

Proof. Let $z = re^{i\theta}$, $a = |a|e^{i\alpha}$, then

$$|1 - z^2/a^2|^2$$

$$= 1 - 2\frac{r^2}{|a|^2}\cos 2(\theta - \alpha) + \frac{r^2}{|a|^4} \geqq 1 - 2\frac{r^2}{|a|^2}\cos 2(1-A)\left(\frac{\pi}{2} - \psi\right) + \frac{r^4}{|a|^2}.$$

Q.E.D.

The following part of the well known Theorem by Titchmarch will be important for the estimation of $F(s)$.

THEOREM A (*Titchmarsh*). *Suppose* $f(z) = \prod_{k=1}^{\infty} (1 - z/a_k)$, *where* $0 < a_k \leqq a_{k+1}$ *for all k and* $\Sigma a_k^{-1} < \infty$; *let* λ, ρ *and* θ *be fixed real numbers satisfying* $\lambda > 0$, $0 < \rho < 1$ *and* $|\theta| < \pi$.

Then $n(x) \sim \lambda x^\rho$ $x \to \infty$ *implies*

$$(2.2) \qquad \log|f(xe^{i\theta})| \sim \pi\lambda \operatorname{cosec} \pi\rho \cos\theta\rho x \qquad x \to \infty$$

where $n(x) = \max\{k \mid a_k \leqq x\}$.

Proof. See for example [5, p. 79].

THEOREM 2.2. *Let the sequence* $\{a_k\}$ *satisfy the condition of class* $C(\rho)$, *then for some* $K > 0$ *and* $M > 0$

$$(2.3) \qquad |F(z)| \geqq K e^{M|z|^{2\rho}}$$

uniformly for $\min_{n=1,3} \left|\arg z - \frac{n\pi}{2}\right| \leqq \eta \equiv \min\left(\frac{\pi}{4} - \frac{\psi}{2}, \frac{\pi}{8\rho} - \frac{\psi}{2}\right)$.

Proof. By Lemma 2.1 we have

$$|F(z)|^2 = |\prod_{k=1}^{\infty}(1 - z^2/a_k^2)|^2 \geqq \prod_{k=1}^{\infty} 1 - 2\frac{|z|^2}{|a_k|^2}\cos 2(1-A)\left(\frac{\pi}{2} - \psi\right) + \frac{r^4}{|a_k|^2}.$$

It is clear that $\eta > 0$ and we can choose A, $0 < A < 1$, such that $\eta = A(\pi/2 - \psi)$ and therefore $2(1-A)\ (\pi/2 - \psi) = 2(\pi/2 - \psi) - 2\eta \equiv 2\beta > 0$ and also $(\pi - 2\beta)\rho = (2\psi + 2\eta)\rho \leqq \pi/2$ (by distinguishing $0 < \rho \leqq \frac{1}{2}$ and $\frac{1}{2} \leqq \rho < 1$ and recalling that in case $\frac{1}{2} \leqq \rho$, $\psi < \pi/4\rho$).

Therefore we have

$$|F(z)|^2 \geqq \prod_{k=1}^{\infty}\left(1 - 2\frac{|z|^2}{|a_k|^2}\cos 2\beta + \frac{r^4}{|a_k|^4}\right) = \left|\prod_{k=1}^{\infty}\left(1 - \frac{r^2}{|a_k|^2}e^{-i2\beta}\right)\right|^2$$

$$= \left|\prod_{k=1}^{\infty}\left(1 + \frac{r^2}{|a_k|^2}e^{i(\pi - 2\beta)}\right)\right|^2 \equiv \left|\prod_{k=1}^{\infty}\left(1 + \frac{R}{|a_k|^2}e^{i(\pi-2\beta)}\right)\right|^2 \equiv f(Re^{i(\pi-2\beta)}).$$

Using Titchmarsh's theorem on $f(z)$ we obtain (since $\operatorname{cosec}\pi\rho > 0$ and $\cos(\pi - 2\beta)\rho > 0$ (for $\pi - 2\beta)\rho < \pi/2$)) $|f(Re^{i(\pi-2\beta)})| \geqq Ke^{MR^\rho}$ and therefore $|F(z)| \geqq Ke^{M|z|^{2\rho}}$ for z in the prescribed angle.

Q.E.D.

THEOREM 2.3. *Suppose the sequence* $\{a_k\}$ *satisfies condition* $C(\rho)$, *then for* $F(s) = \prod_{k1}^{\infty}(1 - s^2/a_k^2)$ *we have*:

(a) $$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\,(F(s))^{-1}\,ds \text{ converges.}$$

(b) $$(-1)^\nu\frac{d^\nu}{dt^\nu}G(-t) = \frac{d^\nu}{dt^\nu}G(t) = \frac{d^\nu}{dt^\nu}\left(\sum_{i=1}^{n} P_i(t)e^{-ta_{k(i)}}\right) + O(e^{-kt}) \qquad t \to \infty$$

where $\operatorname{Re} a_{k(i)} = \min|\operatorname{Re} a_k|$ *and* $p_i(t)$ *are polynomials of order* μ_i *where* $\mu_i + 1$ *is the number of times* $a_{k(i)}$ *appears in* $\{a_k\} \cdot (a_{k(i)} \neq a_{k(j)}$ *if* $i \neq j)$.

(c) $$F(s)^{-1} = \int_{-\infty}^{\infty} e^{-st}G(t)dt.$$

Proof. The proof of (a) follows immediately from Theorem 2.2. A common technique (see [4, p. 108]) yields (b) but we have to use the estimation of Theorem 2.2 to show that $|F(\sigma + i\tau)|^{-1} = O(|\tau|^{-n})$ $|\tau| \to \infty$ (which is readily established). Conclusion (c) is immediate once (b) is proved. It would be perhaps worthwhile to note that J. Dauns and D. V. Widder in a similar theorem [1, p. 432] had not noticed the possibility of many, though finite, different $a_{k(i)}$ with the same real part.

Q.E.D.

3. **The inversion result.** For the inversion result we shall prove the following Lemma.

LEMMA 3.1. *Suppose*:

(1) $f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt$ *is a transform of class* $C(\rho)$.

(2) $\phi(t) \in L_1(A,B)$ *for every* A, B *such that* $-\infty < A < B < \infty$.

(3) $\left|\int_0^t \phi(v)dv\right| \leqq Ke^{M|t|}$ *where* $M < \min|\mathrm{Re}\, a_k| \equiv \alpha_1$.

Then

$$|f(x)| = O(e^{|\alpha_1 x|}) \qquad x \to \infty.$$

Proof. By Theorem 2.3, defining $\alpha(t) = \int_0^t \phi(v)dv$, we have

$$\left|\int_{-\infty}^{\infty} G(x-t)\phi(t)dt\right| = \left|\int_{-\infty}^{\infty} G'(x-t)\alpha(t)dt\right|$$
$$\leqq K\int_{-\infty}^{\infty} |G'(x-t)|e^{M|t|}dt \leqq K\int_{-\infty}^{\infty} |G'(v)|e^{M|x-v|}dv$$
$$\leqq Ke^{M|x|}\int_{-\infty}^{\infty} |G'(v)|e^{M|v|}dv \leqq K_1 e^{M|x|}.$$

Q.E.D.

THEOREM 3.2. *Suppose assumptions* (1), (2) *and* (3) *of Lemma* 3.1 *are satisfied and* (4) $\int_0^h [\phi(x+y) - \phi(x)]dy = o(h)$ $h \to 0$. *Then*

$$\lim_{t\to 0+} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\infty}^{\infty} F(iu)\exp[-tu^2 + iu(x-\xi)du = \phi(x). \tag{3.1}$$

Proof. We first show

$$H(t, x-\xi) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(iu)\exp(-tu^2)\exp(iu(x-\xi))du$$

converges absolutely; this follows immediately if we prove that $F(z)$ is an entire function of order less than 2. Since $|a_k|^2 \leqq x < |a_{k+1}|^2$ for $k \sim \lambda x^\rho$ which implies $x \sim (k/\lambda)^{1/\rho}$ $k \to \infty$ we have $|a_k| \sim (k/\lambda)^{1/2\rho} k \to \infty$ and therefore the exponent of convergence of $\Sigma |a_k|^{-\alpha}$ is 2ρ and by Theorem 14.2.4 of [3, p. 195] $F(z)$ is of order $2\rho < 2$. By Cauchy theorem we have for all B (positive and negative)

$$H(t, x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(B+iu)\exp(t(B+iu)^2\exp(B+iu)(x-\xi)du$$
$$\leqq e^{tB^2}\exp B(x-\xi)\frac{1}{2\pi}\int_{-\infty}^{\infty} F(B+iu)\exp[iu(x-\xi+2tB)]$$
$$\exp(-tu^2)du$$
$$\leqq K_2 e^{tB^2}\exp B(x-\xi).$$

Using different values for B it is clear that for every fixed t the integral $\int_{-\infty}^{\infty} H(t, x-\xi)d\xi$ converges absolutely. We have

$$\int_{-\infty}^{\infty} H(t, x-\xi)d\xi = \left[\int_{-\infty}^{\infty} H(t, x-\xi)e^{-iu(x-\xi)}d\xi\right]_{u=0} = [F(iu)\exp(-tu^2)_{u=0} = 1.$$

Therefore it would be sufficient to show

$$\lim_{t\to 0+} \int_{-\infty}^{\infty} [f(\xi) - \phi(x)]H(t, x-\xi)d\xi \equiv \lim_{t\to 0+} I(t) = 0. \tag{3.2}$$

Obviously the integral in (3.2) converges absolutely for any fixed t using again the estimate of $H(t, x-\xi)$ for different values of B.

We have also

$$\begin{aligned} I(t) &= \int_{-\infty}^{\infty} H(t, x-\xi)d\xi \int_{-\infty}^{\infty} G(\xi - v)[\phi(\xi) - \phi(x)]\,dv \\ I(t) &= \int_{-\infty}^{\infty} H(t, x-\xi)d\xi \int_{-\infty}^{\infty} G'(\xi - v)\alpha(v-x)dv \end{aligned} \tag{3.3}$$

where

$$\alpha(v-x) = \int_x^v [\phi(w) - \phi(x)]dw.$$

By (3) of Lemma 3.1 for a fixed x, $|\alpha(v-x)| \leqq K(x)e^{M|v|}$ and therefore by the method of Lemma 3.1

$$\begin{aligned} \int_{-\infty}^{\infty} |G'(\xi - v)|\,|\alpha(v-x)|\,d\xi &\leqq K(x) \int_{-\infty}^{\infty} |G'(w)|\,e^{M(|w|+|\xi|)}dw \\ &\leqq K(x)\,e^{M|\xi|}\,. \end{aligned}$$

This implies that $I(t)$ converges absolutely and therefore by Fubini's theorem

$$I(t) = \int_{-\infty}^{\infty} \alpha(v-x)dv \int_{-\infty}^{\infty} H(t, x-\xi)G'(\xi - v)d\xi. \tag{3.4}$$

By a well known result on Fourier transforms [7, p. 255]

$$\begin{aligned} &\int_{-\infty}^{\infty} H(t, x-\xi)G'(\xi - v)d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} iu\exp(-tu^2)\exp(iu(x-v))du = -\frac{(x-v)}{\sqrt{4\pi t}\,2t}\exp(-(x-v)^2/4t). \end{aligned}$$

Therefore we have

$$(3.5)\ I(t) = -\left\{\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty}\right\}\alpha(v-x)\frac{(x-v)}{\sqrt{4\pi t}}\cdot\frac{1}{2t}\exp(-(x-v)^2/4t)dv$$
$$\equiv I_1(t) + I_2(t) + I_3(t).$$

It is easily seen by $|\alpha(v-x)| \leqq K(x)e^{M|v|}$ that $I_1(t) = o(1)$ $t \to 0+$ and $I_2(t) = o(1)$ $t \to 0+$ for any fixed $\delta > 0$. Assumption (4) of our theorem yields $|\alpha(v-x)| \leqq \varepsilon |v-x|$ provided $|v-x| < \delta(\varepsilon)$; choosing $\delta = \delta(\varepsilon)$ we have

$$|I_2(t)| \leqq \varepsilon \int_{x-\delta}^{x+\delta} |v-x|^2 t^{-3/2} \frac{1}{\sqrt{\pi}4}\exp(-(x-v)^2/4t)dv$$
$$\leqq \varepsilon \frac{1}{4\sqrt{\pi}} \int_{-\infty}^{\infty} w^2 \exp(-w^2/4)dw \leqq \varepsilon \cdot M$$

since ε is arbitrary (3.4) yields (3.2). Q.E.D.

4. **Remarks.** (1) The special case mentioned by J. Dauns and D. V. Widder [1, p. 442] i.e. $a_k = ke^{i\beta}$ $\beta < \pi/2$ is included in $C(\frac{1}{2})$ since $\max\{k^2 \mid k^2 \leqq x\}$ is when $k \sim x^{1/2}$ $(k \to \infty)$.

(2) In our formula only one limit appears explicitly; the limit as $n \to \infty$ that is explicit in Standish's formula is moved inside the inner integral. This simplifies the proof as well as the appearance of the inversion theorem and in many cases the actual inversion.

References

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